SOLUTION

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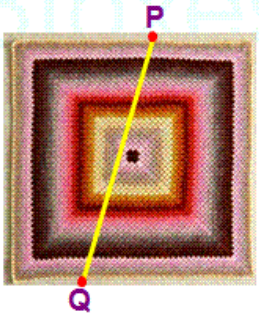
Points P and Q are randomly selected in 2 parallel sides of a square, which is divided into 2 closed areas. What is the expected value of the bigger area to the smaller one?



A) 2
B) e
C) $2\sqrt{2}$
D) 3
E) π

Call x and y the distance of P and Q from their left corners, $0 \le x \le 1$ and $0 \le y \le 1$.

If $x + y \ge 1$, then the left area is the biggest, so the ratio

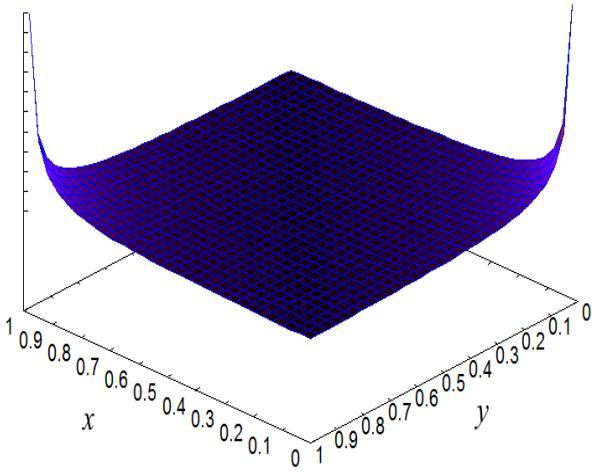
$z = \dfrac{area\ left}{area\ right} = \dfrac{\frac{1}{2}(x+y)}{1-\frac{1}{2}(x+y)} = \dfrac{x+y}{2-(x+y)} = \dfrac{c}{2-c}$, where $c = x + y$.

If $x + y < 1$, then the ratio $z = \dfrac{area\ right}{area\ left} = \dfrac{1-\frac{1}{2}(x+y)}{\frac{1}{2}(x+y)} = \dfrac{2-(x+y)}{x+y} = \dfrac{2-c}{c} = \dfrac{2}{c} - 1$

It's easy to see that the ratio z is symmetrical round $c = 1$:

if $c = 1 + t$, then $z = \dfrac{c}{2-c} = \dfrac{1+t}{2-(1+t)} = \dfrac{1+t}{1-t}$

and if $c = 1 - t$, then $z = \dfrac{2-c}{c} = \dfrac{2-(1-t)}{1-t} = \dfrac{1+t}{1-t}$.



See 3d-plot, where the height is the value of z.

The expected value is the mean height of the surface, and because the ground [0,1]x[0,1] has area 1, is the mean height equal to the volume of the space under the surface. So we need the volume of the space *under* the surface.

Because of the mentioned symmetry this volume can be calculated by:

$$2 \cdot \int_{x=0}^{x=1} \left(\int_{y=0}^{y=1-x} \frac{2}{x+y} - 1 \, dy \right) dx$$

$$= 2 \cdot \int_{x=0}^{x=1} \left(\int_{y=0}^{y=1-x} \left[2\ln(x+y) - y \right]_{y=0}^{y=1-x} \right) dx$$

$$= 2 \cdot \int_{x=0}^{x=1} \left(\int_{y=0}^{y=1-x} 2\ln 1 - (1-x) - 2\ln x \right) dx = 2 \cdot \int_{x=0}^{x=1} (x - 1 - 2\ln x)\, dx = \left[\tfrac{1}{2}x^2 - x - 2(x\ln x - x) \right]_{x=0}^{x=1}$$

$$= 2 \cdot (\tfrac{1}{2} - 1 - 2(0-1)) = 2 \cdot \tfrac{3}{2} = 3$$

So the answer is D.